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NEIGHBORING EXTREMAL GUIDANCE FOR SYSTEMS WITH PIECEWISE LINEAR CONTROL USING TIME AS THE REFERENCE VARIABLE

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to Sharlene,

for her love , support and encouragement

NEIGHBORING EXTREMAL GUIDANCE FOR SYSTEMS WITH PIECEWISE LINEAR CONTROL USING TIME AS THE REFERENCE VARIABLE

by

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THESIS

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May, 1993

Abstract

NEIGHBORING EXTREMAL GUIDANCE FOR SYSTEMS WITH PIECEWISE LINEAR CONTROL USING TIME AS THE REFERENCE VARIABLE

by

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A guidance law for the control of a system in the neighborhood of a nominal suboptimal trajectory is developed. The guidance law is demonstrated using a lunar launch problem with constraints at orbit entry. A set of precomputed gains is used by the guidance law to operate on an extremal path in the neighborhood of the suboptimal trajectory. The guidance law and gains are designed to minimize the change in the desired performance index while still satisfying the final path constraints.

In the lunar launch problem, the nominal suboptimal trajectory minimizes the final time using piecewise linear control. This trajectory is obtained to provide a nominal control history. The guidance law is found by minimizing the

second variation of the suboptimal trajectory performance index subject to the final constraints being satisfied. For the lunar launch problem, the guidance law leads to a set of gains that relates deviations from the suboptimal trajectory to required changes in the nominal control history. The deviations from the suboptimal trajectory, used together with the precomputed gains, determines the change in the nominal control history required to meet the final constraints while minimizing the change in the final time.

Previous research has successfully used this guidance law by referencing the nominal control and corresponding change in the control using horizontal velocity. This research demonstrates the guidance law using time as the reference variable. Insuring that the time-to-go on both the perturbed and nominal trajectories are the same, determines the change in the control and update of the final time that are to be applied to the perturbed trajectory.

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English Symbols

a	parameter vector for suboptimal control problem
G	augmented performance index
$\mathcal{J}$	performance index of neighboring extremal problem
J	performance index
K	gain matrix
k	element of gain matrix
$f(\ )$	state differential equation vector
g	acceleration due to gravity $\left(\frac{ft}{\sec^2}\right)$
$g(\ )$	parameterized state differential equation vector
t	time (sec)
u	horizontal component of velocity $\left(\frac{ft}{\sec}\right)$
v	vertical component of velocity $\left(\frac{ft}{\sec}\right)$
x	state vector
y	altitude (ft)

Greek Symbols

α	thrust acceleration $\left(\frac{ft}{\sec^2}\right)$
Δ	total change
δ	variation
θ	angle between thrust vector and horizontal

ν	vector of Lagrange multipliers used in performance index of suboptimal control problem
τ	normalized time
υ	vector of Lagrange multipliers used in performance index of neighboring extremal problem
Φ	state transition matrix
ϕ	performance index in suboptimal control problem
ψ	equality final constraint vector in suboptimal control problem
Ψ	transition matrix

Subscripts and Superscripts

$(\ )_0$	value at initial time
$(\ )_{0_s}$	specified value at initial time
$(\ )_a$	derivative with respect to the parameters
$(\ )_f$	final value
$(\ )_{opt}$	on the nominal optimal trajectory
$(\ )_p$	on the perturbed trajectory
$(\ )_x$	derivative with respect to the states
$(\ )_{x_f}$	derivative with respect to the states at final point
$(\dot{\ })$	derivative with respect to t
$(\ )'$	derivative with respect to τ
$(\ )^{-1}$	inverse
$(\ )^T$	transpose

Chapter 1

Introduction

Optimal and suboptimal control theory are concerned with finding the optimal control, and consequently the optimal trajectory, for a desired system. The optimal control is found most often using numerical optimization. This method many times leads to a less then optimal solution because the model used to describe the system is not accurate. In addition, the initial conditions or system parameters may not be known exactly, or the equations of motion may have been simplified. As a resu't, when the optimal control is applied to the actual system, deviations from the optimal trajectory occur, and the optimal trajectory is no longer realizable. One solution to this problem is to generate a new optimal trajectory from the current location to the final constraint manifold. However, because of computer limitations, this solution is not feasible for guidance.

Neighboring optimal control is another method used to correct for deviations from the optimal trajectory. Assuming the deviations from the optimal trajectory caused by model errors and/or unknown perturbations are small, an extremal guidance law can be developed to keep the perturbed trajectory in the neighborhood of the optimal or nominal trajectory (see Ref 1, for example).

This research concentrates on developing a neighboring extremal guidance law for systems using piecewise linear control and demonstrating the law with time as the reference variable. A result of using piecewise linear control is that the control is no longer a continuous function but a set of control parameters. These control parameters are the nodes used to define the piecewise linear control. This is also known as suboptimal control. The guidance law, using given deviations from the suboptimal trajectory, updates the control parameters using a set of precomputed neighboring extremal gains. The neighboring extremal gains are determined by finding the change in the control parameters that will minimize the change in the performance index and still satisfy the final constraints.

The guidance law is demonstrated using the lunar launch problem. The lunar launch problem calls for placing a vehicle in a low lunar orbit in minimum time while meeting conditions at orbit entry. Perturbations are caused in the trajectory by altering thrust acceleration or lunar gravity from their nominal values used in solving the nominal suboptimal control problem. Previous work [2] has successfully demonstrated the guidance law using horizontal velocity as the reference variable to determine the nominal control and change in the control. When an attempt is made to implement the guidance law with time as the reference variable, however, problems arise. The lunar launch problem is a free final time problem and the perturbed final time is constantly changing. As a consequence, it is not known exactly where the vehicle is located in relation to the nominal trajectory. Therefore, the nominal control and change in the control that must be applied is not known.

This research uses time as the reference variable, and as a result, the final time becomes another parameter in the control vector. The variation of this parameter yields the change in the final time, which is really the change in the time-to-go from a location on the nominal trajectory specified by the reference time. The change in the time-to-go is caused by deviations from the nominal trajectory at the location specified by the reference time. The overall change in the final time is then given by the difference between the reference time on the nominal optimal trajectory and the current time on the perturbed trajectory such that the time-to-go on both trajectories is the same.

Chapter 2 discusses the general optimal control problem and reformulates the problem in terms of suboptimal control. In Chapter 3, the neighboring suboptimal guidance law is developed. The guidance law is then applied to the lunar launch problem in Chapter 4, with the results shown in Chapter 5. General conclusions and recommendations are discussed in Chapter 6.

Chapter 2

Optimal Control

2.1 Optimal Control Problem

A dynamical system is governed by a set of differential equations

$$\dot{x} = f(t,x,u) \tag{2.1}$$

where x is a n-vector of the state variables and u is a m-vector of the control. The goal is to guide the system from some specified initial conditions

$$t_0 = t_{0_s},\ x_0 = x_{0_s} \tag{2.2}$$

to the prescribed final conditions

$$\psi\left(t_f, x_f\right) = 0, \tag{2.3}$$

by using the control u. Many controls will lead to trajectories that satisfy these boundary conditions, but only one is optimal in terms of a desired performance index. The optimal control, u, is determined by minimizing a performance index in the form

$$J = \phi\left(t_f, x_f\right). \tag{2.4}$$

The general optimal control problem can now be stated as follows: find the control history $u(t)$ that minimizes Eq. (2.4), subject to the differential equations given by Eq. (2.1) and the initial conditions of Eq. (2.2), while satisfying the final constraints given in Eq. (2.3) [1].

The solution to this optimal control problem is usually too complicated to be solved in closed form. The solution to the general optimal control problem may even be too complicated to be solved numerically. Therefore, the general optimal control problem is reformulated as a suboptimal control problem so that a numerical solution can be found.

2.2 Suboptimal Control Problem

In the suboptimal control, or parameter optimization, problem the control is expressed as a parameter vector. This parameter vector contains the required control at different points or nodes along the trajectory. The control between these nodes is found by interpolating the control between appropriate nodes. In addition, a free final time problem can be transformed to a fixed final time problem, and the final time becomes the last parameter.

For this research, consider the class of piecewise linear functions as the set of all possible controls. The one control is expressed as a parameter vector

$$u^T = [u_1, u_2, \cdots, u_k], \tag{2.5}$$

where k is the number of control points or nodes along the trajectory. The control between two control nodes is determined by linear interpolating between the two control nodes. The continuous control function of the optimal control problem is now a set of piecewise linear functions.

Free final time problems are transformed to fixed final time problems by normalizing the variable of integration so that the integration limits are 0 to 1. To normalize the variable of integration, the time is divided by the final time, and the new variable of integration, τ, is defined as

$$\tau = \frac{t}{t_f}. \tag{2.6}$$

The system dynamics can now expressed in terms of the new variable of integration as

$$x' = \frac{dx}{d\tau} = t_f f\left(t_f \tau, x, u\right) = g\left(\tau, t_f, x, u\right), \tag{2.7}$$

where

$$\tau_0 = \tau_{0_s},\ x_0 = x_{0_s},\ \tau_f = 1. \tag{2.8}$$

In this fixed final time suboptimal control problem, the states are a function of the parameterized control, u, and the free final time. Thus, a different parameter vector is defined which includes not only the k control nodes but also the final time. This new parameter vector, a, can be written as

$$a^T = \left[u_1,\ u_2, \cdots,\ u_k,\ t_f\right]. \tag{2.9}$$

With the new parameter vector, the object of the fixed final time suboptimal control problem is to find the piecewise continuous control, where $x_f(a)$ is obtained by integrating the system dynamics

$$x' = \frac{dx}{d\tau} = g(\tau, x, a) \tag{2.10}$$

subject to the boundary conditions

$$\tau_0 = \tau_{0_s},\ x_0 = x_{0_s},\ \tau_f = 1, \tag{2.11}$$

that minimizes the performance index

$$J = \phi\left(a_{k+1}, x_f(a)\right), \tag{2.12}$$

and subject to the boundary constraints

$$\psi\left(a_{k+1}, x_f(a)\right) = 0. \tag{2.13}$$

The optimal control vector, a, can be obtained using a nonlinear programming code such as VF02AD.

Chapter 3

Neighboring Suboptimal Control

The solution to the suboptimal control problem yields a control history and trajectory which minimize the desired performance index. For the suboptimal control history to guide the vehicle to the final boundary constraints and minimize the desired performance index, the vehicle must remain on the nominal trajectory. This assumption is unrealistic, however, because in order for the vehicle to remain on the nominal trajectory the true physical model and the model used to generate the suboptimal control would have to be exactly the same. In reality, the vehicle will not remain on the nominal trajectory because of perturbations in the initial conditions and/or model errors. As a consequence, a guidance law is required that operates the vehicle in the neighborhood of the suboptimal trajectory.

Neighboring suboptimal control is used to develop a guidance law for operating a vehicle in the neighborhood of the suboptimal path. Given that a perturbation away from the desired states occurs, the goal of neighboring suboptimal control is to determine the change in the nominal control required so that the overall change in the performance index is minimized and the boundary conditions are satisfied.

The goal of developing a neighboring suboptimal control law is to find a relationship between deviations from the nominal trajectory and changes required in the nominal control so that the change in the desired performance index is minimized and the boundary conditions are satisfied. The derivation of this control law was originally developed by Hull and Helfrich [3]. A summary of the derivation is included here for completeness.

As shown in Equations (2.12) and (2.13), the value of the performance index and the constraints are a function of the parameter vector a. Therefore, the augmented performance index for the suboptimal control problem is expressed as

$$J' = \phi\left(a_{k+1}, x_f(a)\right) + \nu^T \psi\left(a_{k+1}, x_f(a)\right) = G\left(a_{k+1}, x_f(a), \nu\right) = G(a, \nu), \quad (3.1)$$

where ϕ is the performance index, ψ is the vector of final constraints and ν is the vector of Lagrange multipliers. Given a perturbation in the control vector, a, the change in the performance index is

$$\Delta J' = G(a + \delta a, \nu) - G(a, \nu). \quad (3.2)$$

Since the first variation of the performance index vanishes on the optimal path, the change in the performance index due to a perturbation in the control vector to second order is approximated as

$$\Delta J' = \frac{1}{2!}\delta a^T G_{aa}(a, \nu)\delta a. \quad (3.3)$$

The second derivative matrix G_{aa} is calculated numerically for the optimal path. In Eq. (3.3), the δa' s are not independent but constrained by the requirement that the final constraints be satisfied. Therefore, a relationship is needed to correlate a perturbation in the control, δa, to a deviation from the optimal path, δx_0, such that $\delta\psi = \psi_{x_f}\delta x_f = 0$.

The system dynamics given by Eq. (2.10) relate the state and the control through a set of differential equations. To relate the variation δa to the perturbation δx_0, the variation of Eq. (2.10) is taken as

$$\delta x' = g_x \delta x + g_a \delta a, \quad (3.4)$$

with boundary conditions

$$\tau_0 = \tau_{0_s}, \ \tau_f = 1 \tag{3.5}$$

$$\delta x_0 = \delta x_{0_s} \tag{3.6}$$

$$\psi_{x_f} \delta x_f = 0. \tag{3.7}$$

To solve Eq. (3.4), a solution is assumed to be of the form

$$\delta x = \Phi(\tau,1)\delta x_f + \Psi(\tau,1)\delta a \tag{3.8}$$

with boundary conditions

$$\Phi(1,1) = \mathrm{I} \tag{3.9}$$

$$\Psi(1,1) = 0. \tag{3.10}$$

Substituting the derivative of Eq. (3.8) into the left side of Eq. (3.4), and Eq. (3.8) into the right side of Eq. (3.4) results in

$$(\Phi' - g_x\Phi)\delta x_f + (\Psi' - g_x\Psi - g_a)\delta a = 0. \tag{3.11}$$

To guarantee that Eq. (3.11) is satisfied, the coefficients of δx_f and δa are chosen to be zero. The outcome of setting the coefficients equal to zero is two differential equations defining the transition matrices Φ and Ψ. The equations are

$$\Phi' = g_x\Phi \tag{3.12}$$

and

$$\Psi' = g_x\Psi + g_a \tag{3.13}$$

with boundary conditions

$$\Phi_f = \mathrm{I} \tag{3.14}$$

$$\Psi_f = 0. \tag{3.15}$$

By using Eqs. (3.12) and (3.13), Φ' and Ψ' can be integrated backwards along the nominal trajectory from τ_f to find $\Phi(\tau)$ and $\Psi(\tau)$. Knowledge of the history of the transition matrices allows Eq. (3.8) to be used at any time.

Recall the variation of the system dynamics , given by Eq. (3.4), is subject to the constraint of Eq. (3.7). Multiplying Eq. (3.8) by Φ^{-1} and solving for δx_f yields

$$\delta x_f = -\Phi^{-1}\Psi\delta a + \Phi^{-1}\delta x. \tag{3.16}$$

Applying Eq. (3.16) at τ_0 and substituting this result into Eq. (3.7) leads to

$$\psi_{x_f}\Phi_0^{-1}\delta x_0 - \psi_{x_f}\Phi_0^{-1}\Psi_0\delta a = 0, \tag{3.17}$$

which is a constraint on δa imposed, for a given δx_0, as a result of satisfying the final conditions.

A new optimization problem is now formed to minimize the change in the performance index given by Eq. (3.3) with respect to δa, subject to the constraint of Eq. (3.17). The new augmented performance index is written as

$$\mathcal{J}' = \frac{1}{2!}\delta a^T G_{aa}\delta a + \delta v^T\left(\psi_{x_f}\Phi_0^{-1}\delta x_0 - \psi_{x_f}\Phi_0^{-1}\Psi_0\delta a\right). \tag{3.18}$$

By minimizing Eq. (3.18), the change in the performance index is minimized due to perturbations in the control and subject to the boundary constraint that relates the control and the states.

Taking the first variation of Eq. (3.18) results in

$$\delta\mathcal{J}' = \left(\delta a^T G_{aa} + \delta v^T \psi_{x_f}\Phi_0^{-1}\Psi_0\right)\delta(\delta a). \tag{3.19}$$

To find a minimum, the first variation must vanish. This implies that

$$\delta a^T G_{aa} + \delta v^T \psi_{x_f}\Phi_0^{-1}\Psi_0 = 0 \tag{3.20}$$

needs to be satisfied for a minimum to exist. Transposing Eq. (3.20) and assuming G_{aa}^{-1} exists leads to

$$\delta a = -G_{aa}^{-1}\Psi_0^T\Phi_0^{-T}\psi_{x_f}^T\delta\nu. \tag{3.21}$$

Substituting Eq. (3.21) into Eq. (3.17) yields

$$-\psi_{x_f}\Phi_0^{-1}\Psi_0 G_{aa}^{-1}\Psi_0^T\Phi_0^{-T}\psi_{x_f}^T\delta\nu = \psi_{x_f}\Phi_0^{-1}\delta x_0 \tag{3.22}$$

which can be solved for $\delta\nu$ as

$$\delta\nu = -\left(\psi_{x_f}\Phi_0^{-1}\Psi_0 G_{aa}^{-1}\Psi_0^T\Phi_0^{-T}\psi_{x_f}^T\right)^{-1}\psi_{x_f}\Phi_0^{-1}\delta x_0. \tag{3.23}$$

Inserting $\delta\nu$ into the right side of Eq. (3.21) leads to the neighboring suboptimal control law

$$\delta a = K_0\delta x_0, \tag{3.24}$$

where

$$K_0 = G_{aa}^{-1}\Psi_0^T\Phi_0^{-T}\psi_{x_f}^T\left(\psi_{x_f}\Phi_0^{-1}\Psi_0 G_{aa}^{-1}\Psi_0^T\Phi_0^{-T}\psi_{x_f}^T\right)^{-1}\psi_{x_f}\Phi_0^{-1} \tag{3.25}$$

is the gain matrix at τ_0.

The concept of controllability assumes that to first order a trajectory exists from a perturbed location, δx_0, at a given time, τ, to the final constraint manifold, or $\delta\psi = 0$. In other words, a solution to Eq. (3.4) is desired which satisfies the boundary conditions given by Eqs. (3.5-3.7). Recall that to solve for the suboptimal control gains given in Eq. (3.25), a solution to Eq. (3.4) is assumed in the form of Eq. (3.8). Therefore, the system is assumed to be controllable. This implies there exists at least one control history that will lead from the current perturbed location to the final constraint manifold. The neighboring suboptimal control law determines the one control history that will minimize the change in the performance index while satisfying the final constraints. Thus, the control law given in Eq. (3.24) uses small deviations in the states, δx_0, at a given time, τ, to determine small changes required in the parameters, δa. These small changes are then added to the optimal values of the control parameters, found by solving the suboptimal control problem from τ_0 to τ_f, to form a new control history. This new control history leads to the final constraints while minimizing the change in the performance index.

This method of control can be extended so that any time $\tau_i < \tau_f$, $i = 1,2,\cdots$, may be considered τ_0 or the "initial" time and the control history updated. Because Φ and Ψ are functions of τ only, the gain K_0 can be computed and stored at each node point along the nominal optimal trajectory. When τ_0 occurs between node points, the gain is linearly interpolated. Thus, at each sample time the current states of the vehicle are compared with the states along the nominal optimal path, the appropriate gains chosen, and the change in the control parameters determined.

Chapter 4

Application of the Control Law

4.1 Lunar Launch Problem

The neighboring suboptimal control law is demonstrated using a lunar launch problem. The lunar launch problem involves placing a vehicle into lunar orbit in minimum time (refer to Fig. 4.1). In addition, constraints are placed on the vehicle's position and velocity at orbit entry. The control used to place the vehicle in orbit is the angle of thrust versus time, $\theta(t)$. The lunar launch problem is a free final time problem whose solution provides a control history which minimizes the performance index

$$J = t_f. \tag{4.1}$$

The differential equations governing the trajectory of the vehicle are given by

$$\dot{x} = u \tag{4.2}$$

$$\dot{y} = v \tag{4.3}$$

$$\dot{u} = \alpha \cos\theta \tag{4.4}$$

$$\dot{v} = \alpha \sin\theta - g, \tag{4.5}$$

where α is the thrust acceleration and g is the acceleration due to gravity. The prescribed initial conditions are

$$t_0 = 0,\ x_0 = 0,\ y_0 = 0,\ u_0 = 0,\ v_0 = 0, \tag{4.6}$$

and the final conditions, or the conditions at orbit entry are

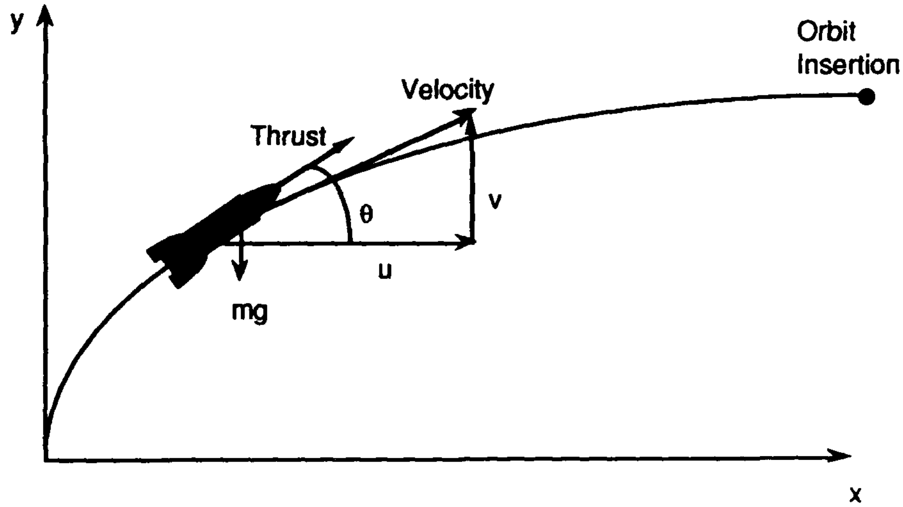

Figure 4.1 The lunar launch problem

$$y_f = 50{,}000 \quad ft \tag{4.7}$$

$$u_f = 5{,}444 \quad ft / \sec \tag{4.8}$$

$$v_f = 0 \qquad ft / \sec . \tag{4.9}$$

The quantities u and v denote the horizontal and vertical velocities, respectively.

To simplify the problem, several assumptions have been made. One assumption, already implied by the equations of motion, is that all motion and thrust act in the x-y plane only. In addition, the vehicle's trajectory is assumed to be over a flat moon, and therefore the direction of lunar gravity remains constant. Also, the magnitude of lunar gravity is assumed to remain constant throughout the trajectory. Finally, the ratio of vehicle mass and thrust is assumed constant throughout the trajectory. This implies that thrust acceleration, α, is constant. Based on previous research on the lunar launch problem [3], values of $g = 5.32 ft / \sec^2$ and $\alpha = 20.8 ft / \sec^2$ are assumed.

The current formulation of the lunar launch problem is as a free final time problem and calls for a continuous control history solution. By using the ideas of suboptimal control, τ is introduced as the new variable of integration and is normalized with respect to the final time. The normalized differential equations become

$$x' = \frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = g\left(\tau, x, \theta, t_f\right). \tag{4.10}$$

The equations of motion for the vehicle are now

$$\dot{x} = t_f u \tag{4.11}$$

$$\dot{y} = t_f v \tag{4.12}$$

$$\dot{u} = t_f \alpha \cos\theta \tag{4.13}$$

$$\dot{v} = t_f(\alpha \sin\theta - g). \tag{4.14}$$

The initial conditions remain

$$x_o^T = [0 \;\; 0 \;\; 0 \;\; 0] \tag{4.15}$$

and the limits of integration become

$$\tau_0 = 0,\; \tau_f = 1. \tag{4.16}$$

To find the suboptimal control history, nine control nodes are evenly spaced throughout the optimal trajectory. The control applied at each node, θ_i, and the optimal final time, t_f, are used to define the suboptimal control, or parameter, vector, a. The optimal control vector, a, is defined as

$$a^T = \left[\theta_1,\; \theta_2, \cdots,\; \theta_9,\; t_f\right]. \tag{4.17}$$

The final conditions, normalized for numerical reasons to have their magnitudes of the same order, are placed in a constraint vector, ψ, where

$$\psi = \begin{bmatrix} \dfrac{y_f}{50000} - 1 \\ \dfrac{u_f}{5444} - 1 \\ \dfrac{v_f}{5444} \end{bmatrix} = 0. \qquad (4.18)$$

The optimal control vector, a, is computed using VF02AD, a nonlinear programming code [4]. The VF02AD program is an extension of a variable metric method for unconstrained optimization applied to constrained optimization. The program uses a recursive quadratic programming variable metric method. A suboptimal solution is searched for until a preset convergence criteria, in this case 10^{-9} sec, is reached. Convergence occurs when the change in the performance index between iterations is less then the preset convergence criteria.

4.2 Calculation of the Neighboring Suboptimal Control Gains

Recall that the gain K_0 is a function of τ and is multiplied by the deviation from the nominal optimal trajectory to determine the required change in the nominal control. Since K_0 is defined as

$$K_0 = G_{aa}^{-1}\Psi_0^T\Phi_0^{-T}\psi_{x_f}^T\left(\psi_{x_f}\Phi_0^{-1}\Psi_0 G_{aa}^{-1}\Psi_0^T\Phi_0^{-T}\psi_{x_f}^T\right)^{-1}\psi_{x_f}\Phi_0^{-1}, \qquad (4.19)$$

to calculate K_0 the differential equations for Φ and Ψ must be solved and the expressions for G_{aa} and ψ_{x_f} evaluated.

To solve for Φ and Ψ, their corresponding differential equations must be integrated backwards from τ_f. The differential equation governing Φ is given by

$$\frac{d\Phi}{d\tau} = \Phi' = g_x\Phi \qquad (4.20)$$

where

$$\Phi_f = I. \tag{4.21}$$

The expression g_x is evaluated as the partial of the system dynamics, given in Eqs. (4.11-4.14), with respect to the states and is given by

$$g_x = \frac{\partial g}{\partial x} = \begin{bmatrix} 0 & 0 & t_f & 0 \\ 0 & 0 & 0 & t_f \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.22}$$

The differential equation governing Ψ is

$$\Psi' = g_x \Psi + g_a \tag{4.23}$$

where

$$\Psi_f = 0. \tag{4.24}$$

In Eq. (4.23), the expression for g_x is given by Eq. (4.22). The expression g_a in Eq. (4.23) represents the partial derivative of the system dynamics with respect to the parameter vector a. Because a is a vector and is expressed as

$$a^T = \left[\theta_1,\ \theta_2, \cdots,\ \theta_9,\ t_f\right], \tag{4.25}$$

the chain rule must be applied to form g_a. Since there are 10 parameters, g_a is written as

$$g_a = \frac{\partial g}{\partial a}\frac{\partial a}{\partial a_i} \qquad 1 \le i \le 10. \tag{4.26}$$

The partial derivatives determine the impact a change in one parameter has on another. The partial derivative relating the change in the system dynamics to the control vector, a, can be expressed by

$$\frac{\partial g}{\partial a} = \frac{\partial g}{\partial \theta} \text{ or } \frac{\partial g}{\partial t_f}. \tag{4.27}$$

The expressions are evaluated as

$$\frac{\partial g}{\partial \theta} = \begin{bmatrix} \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \theta} \\ \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\alpha \sin\theta\, t_f \\ \alpha \cos\theta\, t_f \end{bmatrix} \tag{4.28}$$

and

$$\frac{\partial g}{\partial t_f} = \begin{bmatrix} \frac{\partial x}{\partial t_f} \\ \frac{\partial y}{\partial t_f} \\ \frac{\partial u}{\partial t_f} \\ \frac{\partial v}{\partial t_f} \end{bmatrix} = \begin{bmatrix} u \\ v \\ \alpha \cos\theta \\ \alpha \sin\theta \ \text{-}\ g \end{bmatrix}. \tag{4.29}$$

To calculate the partial derivative of the parameter vector, a, with respect to each element of the parameter vector, a_i, recall the optimal control, given in the first 9 elements of a, is linearly interpolated between adjoining nodes as

$$\theta(\tau) = \theta_{i-1} + \frac{\theta_i - \theta_{i-1}}{\tau_i - \tau_{i-1}}(\tau - \tau_{i-1}), \quad \text{for } \tau_{i-1} \le \tau \le \tau_i \tag{4.30}$$

and

$$\theta(\tau) = \theta_i + \frac{\theta_{i+1} - \theta_i}{\tau_{i+1} - \tau_i}(\tau - \tau_i), \quad \text{for } \tau_i \le \tau \le \tau_{i+1}. \tag{4.31}$$

Therefore, the partial derivatives of Eqs. (4.30) and (4.31) with respect to θ_i, respectively, are

$$\frac{\partial \theta}{\partial \theta_i} = \frac{\tau - \tau_{i-1}}{\tau_i - \tau_{i-1}} \qquad \text{for } \tau_{i-1} \le \tau \le \tau_i \tag{4.32}$$

and

$$\frac{\partial \theta}{\partial \theta_i} = 1 - \frac{\tau - \tau_i}{\tau_{i+1} - \tau_i} \qquad \text{for } \tau_i \le \tau \le \tau_{i+1}. \tag{4.33}$$

Both Eqs. (4.32) and (4.33) are valid for all interior node points. However, each equation breaks down at one of the boundary nodes because either τ_{i-1} or τ_{i+1} does not exist. Therefore, only Eq. (4.33) is used over the first interval and only Eq. (4.32) is used over the last interval. The last element of a is t_f, and the partial of t_f with respect to t_f is one. Over all other intervals, the partial derivative is zero.

The matrix g_a is expressed, using Eqs. (4.28), (4.29), (4.32) and (4.33), as

$$g_a = \begin{bmatrix} 0 & \cdots & 0 & u \\ 0 & \cdots & 0 & v \\ -\alpha \sin\theta \, t_f \dfrac{\partial \theta}{\partial \theta_1} & \cdots & -\alpha \sin\theta \, t_f \dfrac{\partial \theta}{\partial \theta_9} & \alpha \cos\theta \\ \alpha \cos\theta \, t_f \dfrac{\partial \theta}{\partial \theta_1} & \cdots & \alpha \cos\theta \, t_f \dfrac{\partial \theta}{\partial \theta_9} & \alpha \sin\theta - g \end{bmatrix}. \tag{4.34}$$

The derivative, G_{aa}, represents the second derivative effect of a change in the parameter vector on the performance index and can be determined numerically. Central differences are employed, and each element of a is perturbed. The effect of this perturbation on the performance index is computed.

The partial derivative of the constraints with respect to the state vector is represented by ψ_{x_f}. This forms a constant matrix given by

$$\psi_{x_f} = \begin{bmatrix} 0 & \dfrac{1}{50{,}000} & 0 & 0 \\ 0 & 0 & \dfrac{1}{5{,}444} & 0 \\ 0 & 0 & 0 & \dfrac{1}{5{,}444} \end{bmatrix}. \tag{4.35}$$

By using Eq. (4.35), the values computed for G_{aa}, and the solution to the differential equations for Φ and Ψ, the general gain matrix given in Eq. (4.19) can be solved at any control node. These gains, multiplied by the deviation away from the nominal trajectory at the corresponding node point, yield the required change in the control. Because the first column in Eq. (4.35) is all zeros, the first column of the gain matrix will also be all zeros. This is not surprising because the downrange is not constrained, and therefore any deviation in the downrange from the nominal trajectory will not affect the control. This contrasts all of the other states, where any deviation from the nominal trajectory will require a change in the control to insure the final constraints are satisfied. As a result, only the last three columns of the gain matrix will be stored.

Eight gain matrices are computed and correspond to the first eight control nodes. The last control node is at the final constraint, and therefore a gain matrix can not be computed because there are not enough nodes to satisfy the boundary conditions. If the gain matrix associated with the ninth node could be computed, however, it would call for an infinite control change to correct for any deviations from the final constraints. As a result, the gains from the previous two nodes will be used to extrapolate the change in the control over the last interval. The gain matrix, K_i, contains information to change the value of the control at node i and all nodes after node i, given a deviation from the nominal trajectory occurred at node i. The control law will be applied throughout the trajectory, however, and only the gains associated with node i and the node immediately following node i

are needed for applying the control law. Therefore, only these gains will be stored from the gain matrix K_i.

4.3 Interpolating the Change in the Control

The eight gain matrices provide the required changes in the control nodes from node i to the final constraint manifold for a deviation from the nominal trajectory occurring at node i. The change in the control nodes are then added to the nominal controls to provide a new control history. If the deviation from the nominal trajectory occurs between nodes, the gains from two nodes are used and the change in the control linearly interpolated between the two nodes to form a new control history. Previous research by Nowack [2] has successfully used a sample hold technique for interpolating the change in the control between nodes. The nominal control, however, is linearly interpolated throughout the sample step. Therefore, a new method of interpolating the change in the control is implemented which will linearly interpolate the change in the control throughout the sample step.

Assuming a deviation from the nominal trajectory occurred at node i, the change in the control nodes after node i are given by

$$\delta\theta_{i,j} = k_{i,j,y}\delta y_i + k_{i,j,u}\delta u_i + k_{i,j,v}\delta v_i, \tag{4.36}$$

where

i = current node where state deviations occur

j = node where the change in the control is applied

δy_i = deviation in y from nominal trajectory

δu_i = deviation in u from nominal trajectory

δv_i = deviation in v from nominal trajectory and the gain matrix is partitioned as

$$K_i = \begin{bmatrix} k_{i,1,y} & k_{i,1,u} & k_{i,1,v} \\ k_{i,2,y} & k_{i,2,u} & k_{i,2,v} \\ \vdots & \vdots & \vdots \\ k_{i,j,y} & k_{i,j,u} & k_{i,j,v} \end{bmatrix} \begin{bmatrix} \delta y_i \\ \delta u_i \\ \delta v_i \end{bmatrix}. \tag{4.37}$$

The expression $\delta\vartheta_{i,j}$ is only valid when $i \le j$, because it otherwise implies changes to control nodes that have already been passed. Also, notice that all terms involving x have been dropped because any deviation in this state would not affect the control. Realize also that Eq. (4.37) uses deviations in three states to determine the change in the control. This is in contrast to Nowack [2] in which horizontal velocity, u, is used as the reference variable. No deviation, δu_i, ever exists, and therefore only deviations in y and v are used to determine the change in the control.

The change in the control nodes, given in Eq. (4.36), are added to the nominal controls, θ_{0_j}, at each of the remaining nodes. The new control history is formed by linearly interpolating between the new control nodes, θ_j, where

$$\theta_j = \theta_{0_j} + \delta\theta_{i,j}. \tag{4.38}$$

If the deviation from the nominal trajectory occurs between nodes, however, no explicit gain matrix exists and the expression $\delta\theta_{i,j}$ cannot be computed directly. Therefore, the gain matrices corresponding to the control nodes before and after the deviation must be used to interpolate the appropriate change in the control. If the deviation from the nominal trajectory had occurred at node i, a new control history could be determined from node i to the final constraint manifold using $\delta\theta_{i,j}$. Similarly, if the deviation had occurred at node i+1, a different control history from node i+1 to the final constraint manifold

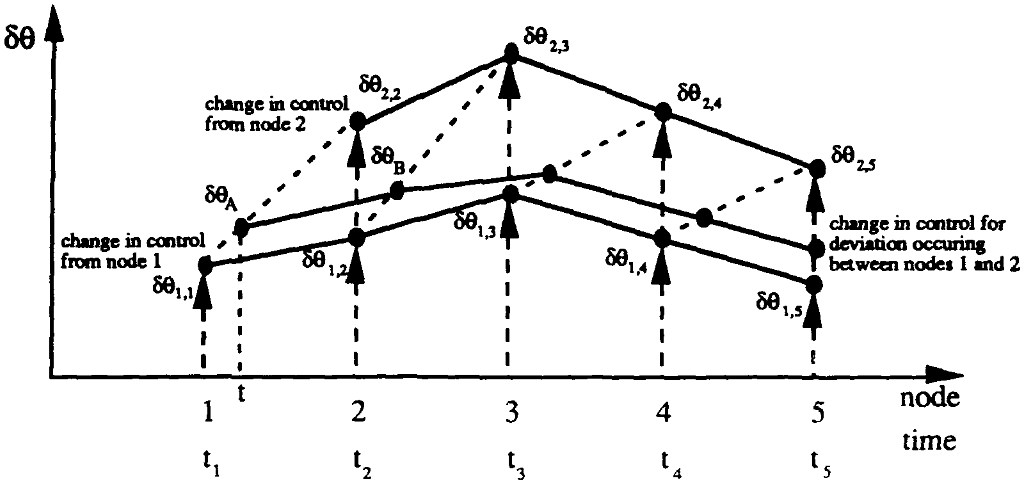

Figure 4.2 Change in the control

could be determined using $\delta\theta_{i+1,j}$. It seems reasonable, therefore, that for a deviation occurring between two nodes there exists *another control history* that lies somewhere between the two control histories. This new control history is likewise determined by using a change in the control that lies somewhere in between $\delta\theta_{i,j}$ and $\delta\theta_{i+1,j}$.

The change in the control for a deviation occurring at time *t*, between nodes 1 and 2 (see Fig 4.2), can be determined by first assuming that the deviation from the nominal trajectory occurred at the previous node, node 1, and by then computing the corresponding change in the control, $\delta\theta_{1,1}$, where

$$\delta\theta_{1,1} = k_{1,1,y}\delta y_1 + k_{1,1,u}\delta u_1 + k_{1,1,v}\delta v_1. \tag{4.39}$$

Now, assume the deviation occurred at the next node, node 2, and compute the change in the control, $\delta\theta_{2,2}$, from node 2 to the final constraint manifold using

$$\delta\theta_{2,2} = k_{2,2,y}\delta y_2 + k_{2,2,u}\delta u_2 + k_{2,2,v}\delta v_2. \tag{4.40}$$

To determine the appropriate change in the control between nodes, the sample hold technique used by Nowack [2] calls for linear interpolating between $\delta\theta_{1,1}$ and $\delta\theta_{2,2}$ to find $\delta\theta_A$. This quantity, given by

$$\delta\theta_A = \delta\theta_{1,1} + \frac{\delta\theta_{2,2} - \delta\theta_{1,1}}{t_2 - t_1}(t - t_1), \tag{4.41}$$

is then held constant throughout the sample step and added to the nominal control. The nominal control, however, is linearly interpolated throughout the sample step. Therefore, to also linearly interpolate the change in the control throughout the sample step, the quantity $\delta\theta_B$ is calculated. This quantity is found by linearly interpolating the change in the control between the control nodes which immediately follow the two nodes where the deviation is assumed to have occurred. In this case the change in the control required at node 2, due to a deviation occurring at node 1, is given by $\delta\theta_{1,2}$. The change in the control required at node 3, due to a deviation occurring at node 2, is given by $\delta\theta_{2,3}$. This quantity $\delta\theta_B$ (see Fig. 4.2) is found by linearly interpolating between $\delta\theta_{1,2}$ and $\delta\theta_{2,3}$ and is given by

$$\delta\theta_B = \delta\theta_{1,2} + \frac{\delta\theta_{2,3} - \delta\theta_{1,2}}{t_3 - t_2}(t - t_1). \tag{4.42}$$

The two quantities $\delta\theta_A$ and $\delta\theta_B$ are now used to define the change in the control required for a deviation from the nominal trajectory occurring at time t between nodes 1 and 2. During the sample step, the change in the control is linearly interpolated between $\delta\theta_A$ and $\delta\theta_B$ as

$$\delta\theta = \delta\theta_A + \frac{\delta\theta_B - \delta\theta_A}{t_2 - t_1}(t - t_1) \tag{4.43}$$

and added to the nominal control to form the new control.

This technique could be used to determine the entire new control history from the point of the deviation to the final constraint manifold, but this is not required. At each sample time, the previous control history would be discarded and a new control history computed. Therefore, the change in the control history is computed only between the two nodes needed to linear interpolate the change in the control.

The last control node is located at the final constraint manifold. There does not exist a gain matrix at this node. As a result, the gain matrices from the previous two nodes are extrapolated over the last interval. The ideas and concepts previously discussed remain valid, but the equations are slightly altered to linearly extrapolate, instead of interpolate, over the last interval.

4.4 Calculation of the Reference Time

All the expressions are now known to solve the gain matrix at each node. The interpolation of the gain matrices between nodes to determine the change in the control given a specific reference time, t_0, is also understood. It is not known, however, at what specific time, t_0, the gains are to be interpolated. Recall that the gains, as well as the nominal control, are evaluated with respect to τ and that τ is related to both a time on the nominal trajectory and the optimal final time, $t_{f_{opt}}$. Assuming the initial conditions are the same at the beginning of the trajectory, the final time of the perturbed trajectory is assumed the same as the optimal final time of the nominal trajectory. But, as soon as some deviation from the nominal trajectory occurs, the optimal final time is no longer realizable. If the current time on the perturbed trajectory is used to determine the appropriate gain to apply, the obvious problem arises when the final time of the perturbed trajectory is greater than the final time of the nominal trajectory [5,6]. The gains and the nominal control are known only until $\tau = 1$, which corresponds to a time equal to $t_{f_{nom}}$. Therefore, to insure that both gains and a nominal control are available throughout the trajectory, a method which incorporates time-to-go is used to reference both the nominal control and the gains associated with the change in the nominal control.

The lunar launch problem is a free final time problem whose nominal solution yields an optimal final time, control history, and trajectory to the final constraint manifold. The controls are applied by determining the location of the vehicle on the nominal trajectory with respect to τ. If the vehicle remains on the nominal trajectory, the final time does not change, and the current time is used to reference the control. When the vehicle deviates from the nominal trajectory, however, the final time changes and the location of the vehicle with respect to the nominal trajectory is unknown. Thus, it is not known what gains should be used to determine the change in the control, or even what nominal control should be applied. Therefore, using the ideas of Kelley [5] and Speyer and Bryson [6], a location on the nominal trajectory is found such that the time-to-go both on the nominal trajectory and the perturbed trajectory is the same. This provides a location of the vehicle with respect to the nominal trajectory and a reference time to determine the nominal control and the change in the control. Having both the nominal and perturbed trajectories always referenced at the same time-to-go guarantees that a nominal control and change in the control will be available throughout the entire trajectory, regardless of the perturbed final time.

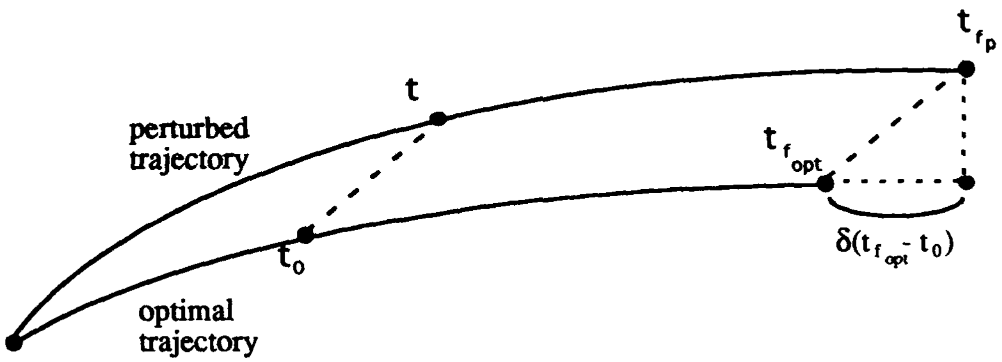

Figure 4.3 Relating time-to-go

The change in the final time caused by a deviation from the nominal trajectory is expressed as

$$t_{f_p} - t_{f_{opt}} = t - t_0, \tag{4.44}$$

where t_{f_p} is the perturbed final time, $t_{f_{opt}}$ is the optimal final time on the nominal trajectory, t is the actual time on the perturbed trajectory and t_0 is a reference time on the nominal trajectory (see Fig 4.3). Equating time-to-go on the perturbed and nominal trajectories results in

$$t_{f_p} - t = t_{f_{opt}} - t_0 + \delta\left(t_{f_{opt}} - t_0\right), \tag{4.45}$$

where $\delta\left(t_{f_{opt}} - t_0\right)$ is the difference between time-to-go on the perturbed trajectory and the time-to-go on the optimal trajectory at the reference time, t_0. Recall the last element of the parameter vector, a, is defined as t_f. The variation of this expression defines the change in the final time, where the final time is the time from the reference time, t_0, to the optimal final time, $t_{f_{opt}}$. The gains used at the reference time, t_0, were evaluated by integrating the transition matrices, given by Eqs. (3.12-3.15), backwards from τ_f, or $t_{f_{opt}}$, to the control node where the gains are stored. The control nodes are identified in each integration as τ_0, but since the optimal final time is known, the control nodes could also be referenced by their corresponding reference time, t_0. Thus, δt_f relates a change in time, not over the entire trajectory, but from the reference time, t_0, to the end. Therefore, the quantity δt_f can be written as $\delta\left(t_{f_{opt}} - t_0\right)$, or the change in the time-to-go from the nominal trajectory to the perturbed trajectory.

The goal is to find a reference time, t_0, where the time-to-go on each trajectory is the same. This implies that δt_f is equal to zero. If t_0 is the reference time where the time-to-go on both trajectories is the same, then a time earlier than t_0 will yield a positive δt_f and a time after t_0 a negative δt_f (see Fig. 4.3). Thus, a one-dimensional search routine is used to search for a reference time where δt_f is zero. Because the routine searches for a minimum, the input for the search is the reference time from the previous step and a search is made over $\left(\delta t_f\right)^2$. The output of the one-dimensional search is a new reference time on the nominal trajectory. This new reference time corresponds to the new time on the perturbed trajectory and indicates where δt_f is zero. This new reference time is then used to determine the nominal control and the change in the control required to satisfy the final constraints. The difference in the actual time on the perturbed trajectory, t, and the new reference time, t_0, furnishes a new estimate of the perturbed final time using Eq. (4.44).

Forcing the time-to-go on the perturbed trajectory to be the same as the time-to-go on the nominal path assumes that there are no further perturbations in the system. If the previous perturbations in the system are due to errors in the model, they will not be accounted for in the new control history. The new control history is based on the same model used to generate the optimal control vector. Thus, the actual perturbed final time is not known until it is actually reached. Throughout the trajectory, the process continues to update the perturbed final time and moves it closer to the actual final time.

4.5 Simulation

A simulation using the lunar launch problem is made to test the guidance law. The differential equations of motion for the vehicle are integrated using a fourth-order Runge-Kutta integrator and a 4 second sample step. Deviations from the nominal trajectory are introduced by perturbing the values of thrust acceleration or lunar gravity $\pm 5\%$ from their nominal values. The nominal trajectory is integrated to the appropriate reference time and deviations in the states are measured. The simulations are accomplished closed loop using both the linear interpolated control and change in the control. The results are compared with simulations obtained by Nowack [2] in which the reference variable was changed to horizontal velocity.

Chapter 5

Results

5.1 Nominal Suboptimal Trajectory

The nine node suboptimal trajectory and corresponding final time are first calculated using the nonlinear programming code VF02AD. The result of this yields the following control history:

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \\ \theta_7 \\ \theta_8 \\ \theta_9 \end{bmatrix} = \begin{bmatrix} 0.45427200240 \\ 0.40895291023 \\ 0.36254811487 \\ 0.31389511079 \\ 0.26421940074 \\ 0.21307434261 \\ 0.16035307712 \\ 0.10711091552 \\ 0.05217078106 \end{bmatrix} \text{rad} = \begin{bmatrix} 26.02786848706 \\ 23.43127577463 \\ 20.77247685115 \\ 17.98486505714 \\ 15.13865652683 \\ 12.20826055367 \\ 9.18755455052 \\ 6.13700339899 \\ 2.98916556845 \end{bmatrix} \text{deg.} \tag{5.1}$$

The vehicle follows the nominal trajectory using the control history given in Eq. (5.1) and reaches orbit with the final conditions:

$$t_{f_{opt}} = 272.70605 \text{ sec} \tag{5.2}$$

$$x_f = 729{,}138.05373 \text{ ft} \tag{5.3}$$

$$y_f = 50{,}000.00000 \text{ ft} \tag{5.4}$$

$$u_f = 5{,}444.00000 \text{ ft/sec} \tag{5.5}$$

$$v_f = 0.00000 \text{ ft/sec.} \tag{5.6}$$

5.2 Neighboring Suboptimal Control Gains

The gain calculations are made using the nominal control history given in Eq. (5.1). The gain program first integrates the equations of motion to the final constraint manifold and then integrates backwards to each of the control nodes. Only certain control gains are required to be stored, because the guidance law is applied throughout the trajectory. These correspond to gains associated with the change in the control made at the node where the deviation occurred, the gains associated with the change in the control made at the following node, and the gains associated with the change in the time-to-go. The result of the gain calculations for the change in the control are given in Table 5.1. For a deviation occurring at each node, the gains associated with the change in the control at that node are shown in the first row, while the second row shows the gains associated with a change in the control at the following node. All gains associated with x are zero and have been omitted from the table. The gains associated with the change in the time-to-go are given in Table 5.2, where again the gains associated with x are zero and have been omitted.

An obvious difference in the gains presented by Nowack [2] and the gains presented in Tables 5.1-5.2 is the number of gains. When horizontal velocity is the reference variable, it is not a state and no deviation in horizontal velocity exists from the nominal trajectory at the update point. Therefore, there is no gain associated with a deviation in horizontal velocity. In addition, time has been included in the state vector instead of horizontal velocity. Because time has no impact on satisfying the constraints, the gains associated with time are zero and

NODE	NODE GAINS		
	y	u	v
1	-0.369E-5	0.679E-4	-0.673E-3
	-0.294E-5	0.539E-4	-0.575E-3
2	-0.284E-5	0.172E-4	-0.456E-3
	-0.462E-5	0.866E-4	-0.785E-3
3	-0.372E-5	0.205E-4	-0.510E-3
	-0.617E-5	0.978E-4	-0.911E-3
4	-0.547E-5	0.321E-4	-0.623E-3
	-0.842E-5	0.107E-3	-0.106E-2
5	-0.888E-5	0.521E-4	-0.806E-3
	-0.120E-4	0.114E-3	-0.125E-2
6	-0.168E-4	0.893E-4	-0.114E-2
	-0.174E-4	0.104E-3	-0.149E-2
7	-0.429E-4	0.178E-3	-0.194E-2
	-0.194E-4	0.709E-5	-0.159E-2
8	-0.248E-3	0.583E-3	-0.560E-2
	0.247E-3	-0.109E-2	0.271E-2

Table 5.1 Control gains associated with a change in the control

not presented by Nowack. On the other hand, three gains are required when time is the reference variable. These gains are required to account for deviations from the nominal trajectory in each of the constrained states. Only x in the state vector does not impact the final constraints, and therefore only the gains associated with x are zero.

5.3 Trajectory Simulation

Simulations using the guidance law are compared with results obtained by Nowack [2]. Tables 5.3 and 5.4 show results when perturbations in thrust acceleration or lunar gravity are added into the system. Each table lists the cause

NODE	NODE GAINS		
	y	*u*	*v*
1	-0.776E-7	-0.487E-4	-0.237E-4
2	-0.887E-7	-0.556E-4	-0.241E-4
3	-0.104E-6	-0.649E-4	-0.246E-4
4	-0.124E-6	-0.779E-4	-0.253E-4
5	-0.156E-6	-0.973E-4	-0.263E-4
6	-0.208E-6	-0.130E-3	-0.281E-4
7	-0.313E-6	-0.195E-3	-0.315E-4
8	-0.632E-6	-0.389E-3	-0.418E-4

Table 5.2 Control gains associated with time-to-go

of the perturbation as well as the optimal final time if the cause of the perturbation had been assumed when computing the suboptimal trajectory. For each perturbation, the first row shows the deviation in the final values of each of the constrained states. The second row shows the same deviations for simulations accomplished using the guidance law presented by Nowack [2].

The actual optimal final time, shown in Tables 5.3 and 5.4, is found using the same optimization code that solved for the nominal control and optimal final time. The optimization code determines the actual final time when the nominal value for thrust acceleration or lunar gravity is replaced by the perturbed value for thrust acceleration or lunar gravity. The perturbed final times of both guidance laws are close to the actual optimal final time. The difference in the perturbed final time and the actual final time is under 0.1 seconds in all cases.

The results of the simulation also show no large disparity between guidance laws in terms of meeting endpoint constraints. Both laws handle deviations caused by thrust acceleration perturbations better, as evidenced by smaller position errors. Velocity errors are approximately the same regardless of the cause of the deviation or the guidance law used.

A comparison of the guidance laws throughout the trajectory is presented in Figs. 5.1-5.8. Figures 5.1 and 5.4 show the deviation in vertical velocity of the perturbed trajectory from the nominal trajectory when the control updates are made for each of the guidance laws. The perturbed trajectory remains closer to the nominal trajectory with time as the reference variable, regardless of the perturbation in the system. The deviations in vertical velocity, shown in Figs. 5.2 and 5.5, show only a slight improvement with time as the reference variable and when the deviations are caused by perturbations in lunar gravity. The maximum deviation in vertical velocity caused by errors in thrust acceleration are slightly reduced with time as the reference variable. However, deviations from the nominal trajectory are not always smaller with time as the reference variable than when horizontal velocity is the reference variable.

The deviation in horizontal velocity from the nominal trajectory is zero when horizontal velocity is the reference variable. Therefore, it is not shown in Figs. 5.3 and 5.7. The magnitude of the deviations in horizontal velocity with time as the reference variable, however, is smaller than deviations in vertical velocity.

Figures 5.4 and 5.8 show the norm of the error of the three constrained states throughout the trajectory. Both guidance laws have similar norms throughout the trajectory, regardless of the cause of the deviation. The norm with time as the reference variable is slightly smaller throughout the trajectory when the deviations are caused by perturbations in lunar gravity. Perturbations in thrust accelerations lead to smaller norms overall. However, neither guidance law yields a smaller norm throughout the trajectory. Both guidance laws show an obvious increase in the norm when the last control node is passed.

α $\left(\frac{ft}{sec^2}\right)$	% change in α (actual final time)	Reference Variable	Perturbed Final Time (sec)	Final Deviation in States y	u	v
19.760	-5	t	287.957	8.958	0.241	-2.868
	(287.950)	u	287.947	10.530	0.0	-2.821
19.968	-4	t	284.764	7.724	0.192	-2.296
	(284.759)	u	284.756	8.931	0.0	-2.263
20.176	-3	t	281.645	6.209	0.143	-1.716
	(281.642)	u	281.639	7.086	0.0	-1.703
20.384	-2	t	278.596	4.445	0.096	-1.152
	(278.595)	u	278.593	4.988	0.0	-1.140
20.592	-1	t	275.619	2.390	0.049	-0.588
	(275.618)	u	275.616	2.630	0.0	-0.573
21.008	+1	t	269.857	-2.664	-0.052	0.588
	(269.859)	u	269.860	-2.909	0.0	0.579
21.216	+2	t	267.070	-5.695	-0.124	1.200
	(267.073)	u	267.076	-6.111	0.0	1.167
21.424	+3	t	264.343	-9.078	-0.155	1.824
	(264.347)	u	264.351	-9.615	0.0	2.369
21.632	+4	t	261.672	-12.783	-0.254	2.434
	(261.679)	u	261.684	-13.435	0.0	2.369
21.840	+5	t	259.053	-16.990	-0.418	3.127
	(259.068)	u	259.073	-17.582	0.0	2.987

Table 5.3 Results for errors in modeling thrust acceleration

g $\left(\frac{ft}{sec^2}\right)$	% change in g (actual final time)	Reference Variable	Perturbed Final Time (sec)	Final Deviation in States y	u	v
5.054	-5	t	271.838	75.079	0.175	-3.333
	(271.780)	u	271.850	65.178	0.0	-2.705
5.107	-4	t	271.999	60.051	0.139	-2.656
	(271.961)	u	272.005	52.081	0.0	-2.160
5.160	-3	t	272.166	45.022	0.105	-1.996
	(272.144)	u	272.168	39.009	0.0	-1.616
5.214	-2	t	272.339	29.987	0.070	-1.332
	(272.329)	u	272.340	25.968	0.0	-1.074
5.267	-1	t	272.520	14.979	0.035	-0.667
	(272.516)	u	272.519	12.963	0.0	-0.536
5.373	+1	t	272.899	-14.909	-0.035	0.664
	(272.898)	u	272.901	-12.914	0.0	0.532
5.426	+2	t	273.098	-29.784	-0.070	1.328
	(273.092)	u	273.105	-25.775	0.0	1.060
5.480	+3	t	273.305	-44.608	-0.105	1.989
	(273.289)	u	273.316	-38.578	0.0	1.584
5.533	+4	t	273.517	-59.380	-0.140	2.648
	(273.488)	u	273.535	-51.317	0.0	2.103
5.586	+5	t	273.737	-74.098	-0.174	3.302
	(273.690)	u	273.762	-63.989	0.0	2.616

Table 5.4 Results for errors in modeling lunar gravity

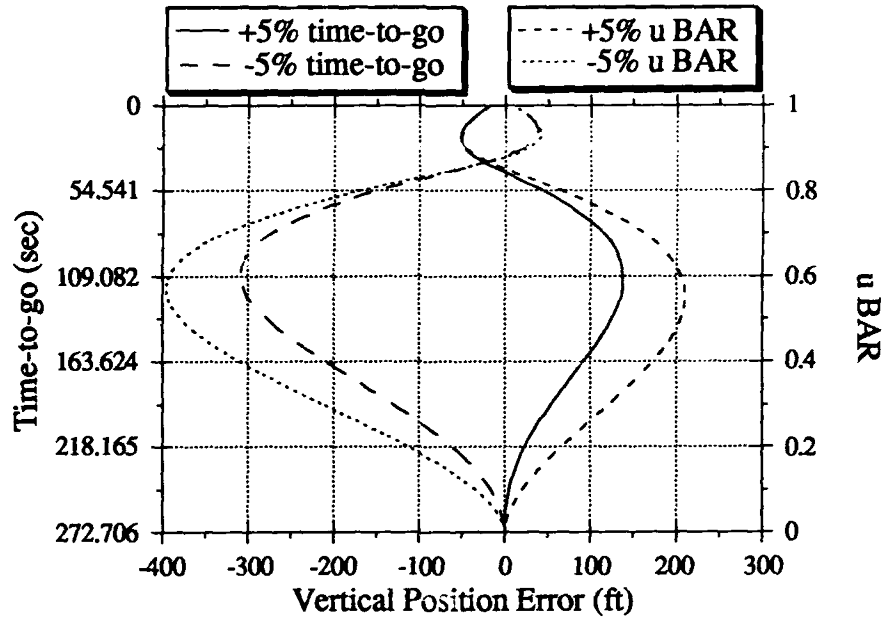

Figure 5.1 Deviation in vertical position for a thrust acceleration perturbation

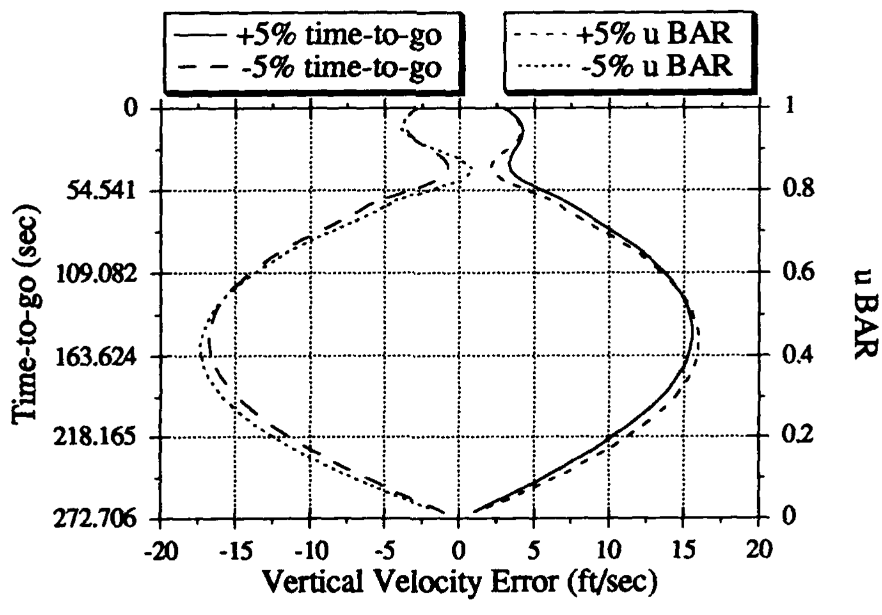

Figure 5.2 Deviation in vertical velocity for a thrust acceleration perturbation

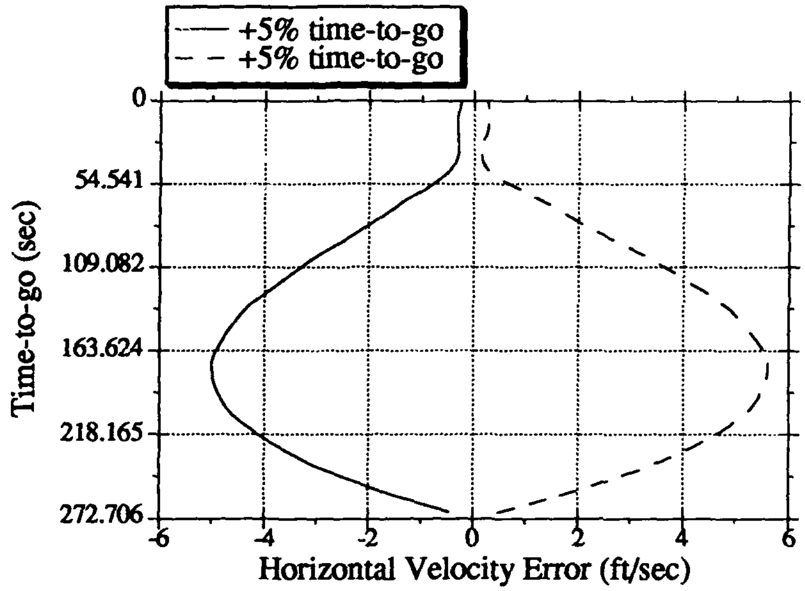

Figure 5.3 Deviation in horizontal velocity for a thrust acceleration perturbation

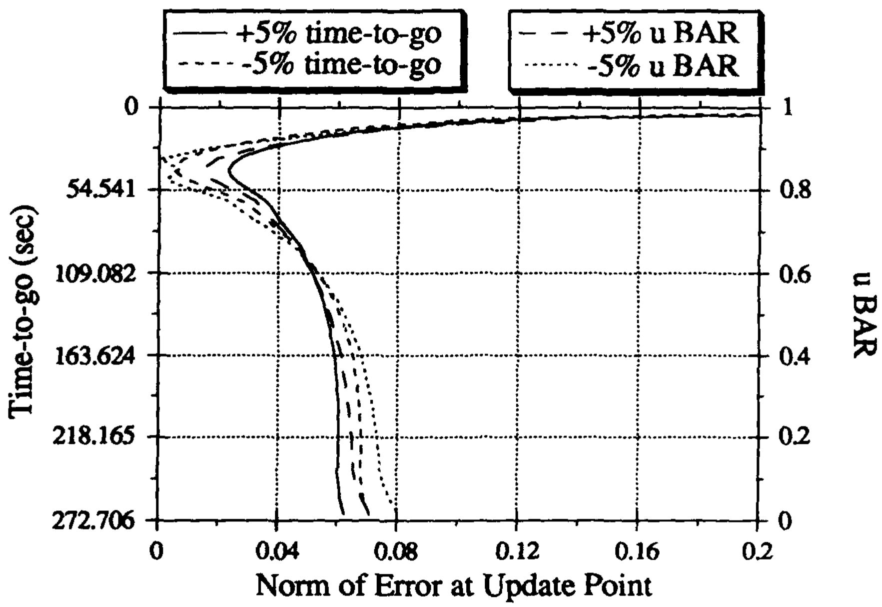

Figure 5.4 Norm of error for a thrust acceleration perturbation

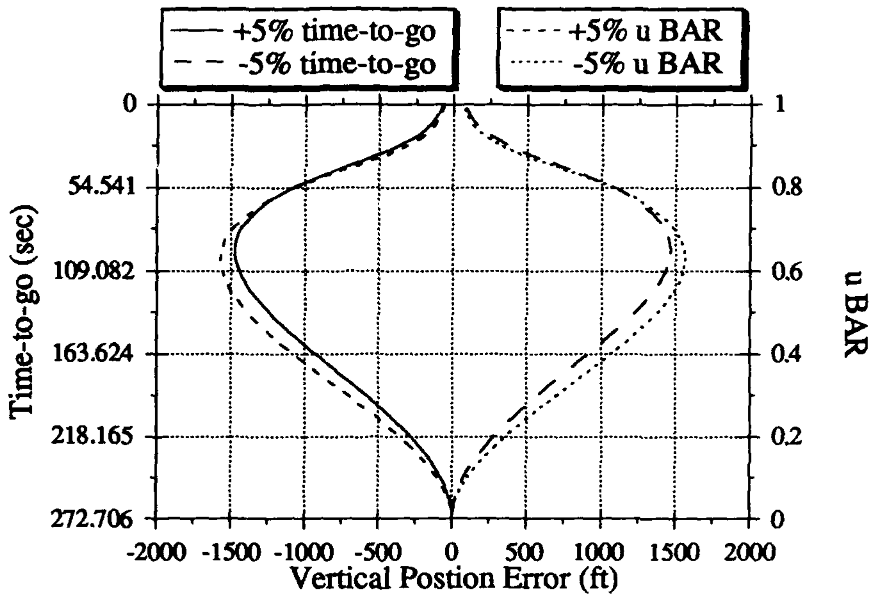

Figure 5.5 Deviation in vertical position for a lunar gravity perturbation

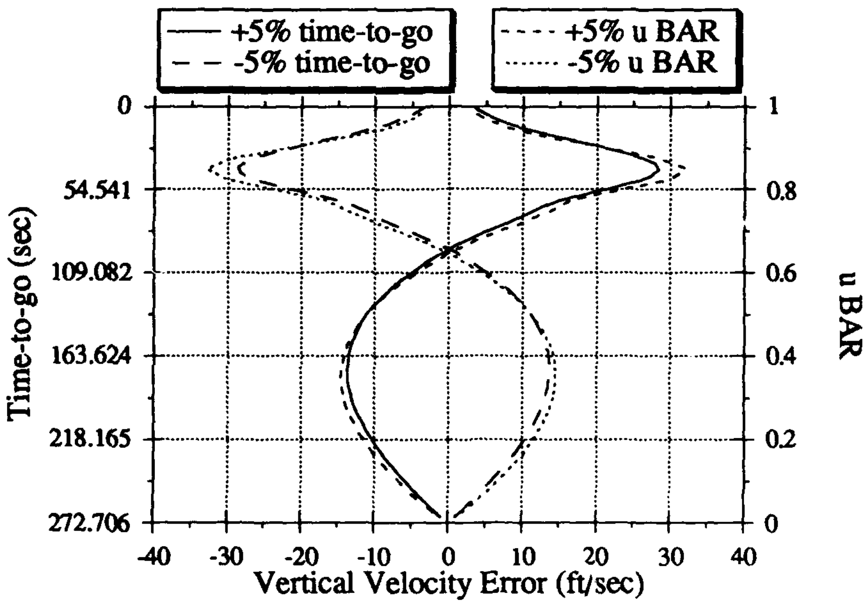

Figure 5.6 Deviation in vertical velocity for a lunar gravity perturbation

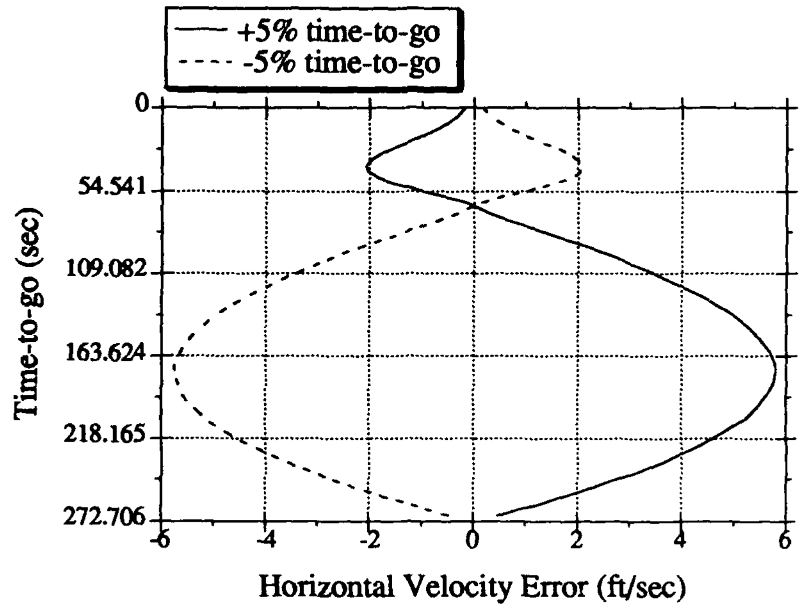

Figure 5.7 Deviation in horizontal velocity for a lunar gravity perturbation

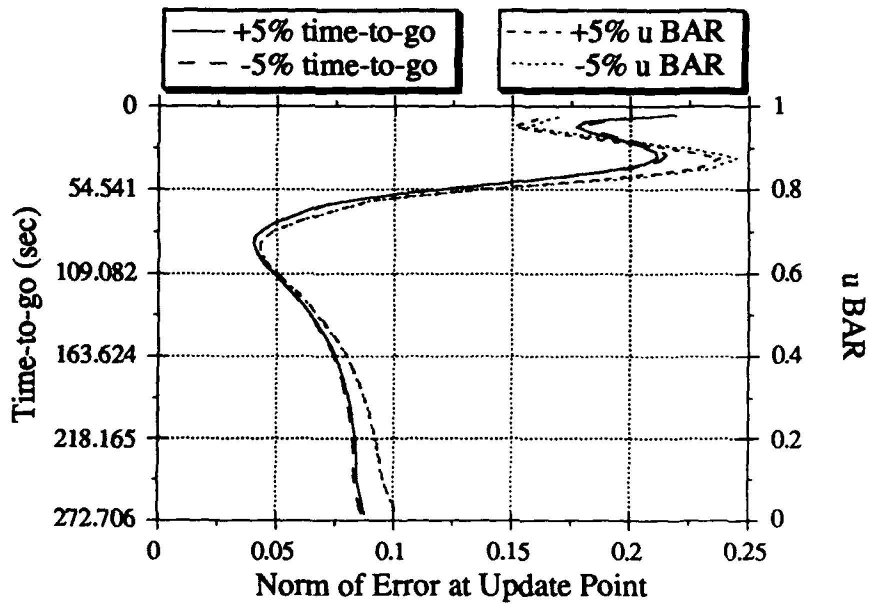

Figure 5.8 Norm of error for a lunar gravity perturbation

Chapter 6

Conclusions

The purpose of this research is to demonstrate the neighboring extremal guidance law using time as the reference variable. The use of horizontal velocity as the reference variable has been previously demonstrated by Nowack [2]. By using current time as the reference variable, the location of the vehicle on the perturbed trajectory is unknown, since the final time is continually changing. Therefore, a one-dimensional search is used to locate the vehicle on the perturbed trajectory by relating time-to-go on the perturbed trajectory to the same time-to-go on the nominal trajectory. This provides the appropriate reference time to determine the nominal control and the change in the control.

The guidance law, with time as the reference variable, is demonstrated using the lunar launch problem. The lunar launch problem involves placing a vehicle into lunar orbit in minimum time while insuring that certain orbit entry parameters are satisfied. The perturbed trajectory is integrated using equations of motion that have been altered from those equations of motion used to generate the nominal trajectory. These altered equations of motion are determined by changing the values of thrust acceleration or lunar gravity. The states at each sample time are then compared to the states on the nominal trajectory. A reference time is found on the nominal trajectory such that the time-to-go on both trajectories is the same. The nominal control at this reference time and the change in the nominal control caused by deviations in the states are used to determine the new control history.

The results of this simulation are compared with results obtained using horizontal velocity as the reference variable. In all cases, the results are comparable. When thrust acceleration is perturbed and time is used as the reference variable, the final value of the states differ from Nowack's [2] results by less then 2 feet in position and by less the 0.5 ft/sec in velocity. The differences are slightly larger, 10 feet and 1 ft/sec, when the perturbations are caused by errors in lunar gravity.

Both laws display a noticeable growth in the norm of the error after the last node point is reached. This is expected, because the change in the control over the last interval is an extrapolation from the previous interval. This leads to an obvious conclusion that meeting the final constraints exactly is an impossibility. With this is mind, no definition of an acceptable error has been given, making it difficult to determine how well the guidance law behaves. One definition, used by Helfrich [3], has the criterion that the percentage error in the final states should not exceed the percentage error in the perturbations. This criteria seems too restrictive when the final constraint is zero. Therefore, realistic problems need to be used in which realistic and acceptable errors are defined.

Areas of further research include understanding time-to-go and its relationship to interpreting the change in the control. This research has tacitly assumed, and not satisfactorily explained, a relationship between time-to-go and the determination of the appropriate change in the control. However, it has demonstrated through results that a relationship exists. Also, an investigation of better terminal guidance is required. Either increasing the number of control nodes or shifting the nodes closer to the final constraints may increase the accuracy of the terminal phase of the trajectory. This research has used linear extrapolation over the last interval, but it is known that the gains increase to infinity as the final constraints are reached. Therefore, a different method of extrapolating the gains over the last interval might yield better results.

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Vita

William Abram Libby was born July 22, 1968 in Winston Salem, North Carolina, son of William Arthur Libby and Sally Meissner Libby. He graduated from Los Alamitos High School in 1986. Mr. Libby graduated from the United States Air Force Academy in 1990 with a degree in Astronautical Engineering and a commission as a second lieutenant in the United States Air Force. He entered Undergraduate Pilot Training in 1990 and graduated a year later. Mr. Libby began his masters program at the University of Texas, Austin in January 1992.

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